Gluing algebras to points

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$$X = SP^1(X) \to SP^2(X) \to SP^3(X) \to \dots$$

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By counting multiplicities, you can write elements $[x_1, \dots, x_n]$ as $\sum_{x \in X \setminus \{x_0\}} m_x x$ with $m_x \in \mathbb{N}$ and $m_x = 0$ for almost all $x \in X$.

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Dold and Thom show: $\pi_i(SP(X), [x_0]) \cong H_i(X; \mathbb{Z})$ for $i > 0$, if X is a connected CW complex.

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so m codifies a commutative multiplication. Note that m is also associative.

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Here, $b = \sum_{i=0}^{n} (-1)^i d_i$ where
$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_n$ for $i < n$ and
$d_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1}$.

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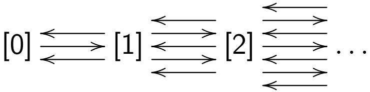

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$d_i\colon [n] \to [n-1]$,

$$d_i(j) = \begin{cases} j, & j < i \\ i, & j = i < n, \quad (0, \quad j = i = n), \\ j-1, & j > i. \end{cases}$$

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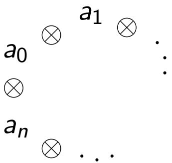
The circle had a cyclic ordering of the points, so A could be taken to be associative:



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Definition Let X be a finite simplicial set and let $R \to A$ be a map of commutative rings, then the *Loday construction of A over X relative R* is

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If $f\colon [m] \to [n] \in \Delta$, then the induced map $f^*\colon \mathcal{L}_X(R)_n \to \mathcal{L}_X(R)_m$ is given by $f^*(\bigotimes_{x \in X_n} r_x) = \bigotimes_{y \in X_m} b_y$ with $b_y = \prod_{f(x)=y} r_x$ where the product over the empty set is defined to be $1 \in R$.

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with $b_y = \prod_{f(x)=y} r_x$ where the product over the empty set is defined to be $1 \in R$.
The definition goes back to Pirashvili, 2000.

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  For any two finite simplicial sets X and Y we always get
  $$\mathcal{L}^R_{X \times Y}(A) \cong \mathcal{L}^R_X(\mathcal{L}^R_Y(A)).$$
  So one can view torus homology as iterated (topological) Hochschild homology.

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So iterating K-theory produces interesting objects.

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Theorem [Dundas-Lindenstrauss-R 2018; Mandell]
For all $n \geq 2$:

$$\pi_* \mathcal{L}_{S^n}(\mathbb{F}_p) \cong \mathrm{Tor}_{*,*}^{\pi_* \mathcal{L}_{S^{n-1}}(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

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What if it just depended on the homotopy type of ΣX?

As there is a homotopy equivalence

$$\Sigma T^n \simeq \Sigma(\bigvee_{k=1}^{n} \bigvee_{\binom{n}{k}} S^k)$$

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Lindenstrauss-R, 2022: Thom spectra associated to Ω^∞-maps are stable, (real and complex) topological K-theory is stable and $HR \to HR/(a_1, \ldots, a_n)$ is stable if R is a commutative ring and the sequence $(a_1, \ldots, a_n)$ is regular.

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Need to work with ∞-categories: Objects are n-manifolds as above. The morphism space from M_1 to M_2 is the space of embeddings: Mfd_n.

Then the ∞-category of n-manifolds with B-framing, Mfd_n^B, is defined as the pullback:

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Idea of *factorization homology*: Take algebras, that can digest 'disks' and average these algebras over M.

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Definition For $A \in \mathrm{Fun}^{\otimes}(\mathrm{Disk}_n^B, \mathcal{C})$ the *factorization homology of A over M* is

$$\int_M A := Y_M \otimes_{\mathrm{Disk}_n^B} A$$

where Y_M is the Yoneda functor sending $\bigsqcup_{i=1}^n \mathbb{R}^n$ to $\mathrm{Mfd}_n^B(\bigsqcup_{i=1}^n \mathbb{R}^n, M)$.

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People involved: Lurie, Ayala-Francis, Klang, Andrade, Rozenblyum, Costello, Gwilliam, Scheimbauer, Gaitsgory, Tanaka, ...

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$$\int_{\Sigma_g} H\mathbb{F}_2 = H\mathbb{F}_2 \wedge (S^3 \times (\Omega S^3)^{2g})_+.$$